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$\mathcal{N}=4$ SYM on K3 and the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence

Kazumi Okuyama

Department of Physics, Shinshu University,
Matsumoto 390-8621, Japan
E-mail: kazumi@azusa.shinshu-u.ac.jp

Abstract: We study the Fareytail expansion of the topological partition function of $\mathcal{N}=4$ SU(N) super Yang-Mills theory on K3. We argue that this expansion corresponds to a sum over geometries in asymptotically AdS_3 spacetime, which is holographically dual to a large number of coincident fundamental heterotic strings.

Keywords: Supersymmetric gauge theory, Superstrings and Heterotic Strings, AdS-CFT Correspondence.

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Contents

1. Introduction

The AdS/CFT correspondence is a powerful way to study the quantum gravity with a negative cosmological constant. In particular, the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence is interesting from the viewpoint of quantum gravity since three dimensional gravity has no propagating degrees of freedom at the classical level, hence the bulk theory might be simpler than the higher dimensional cousins. Recently, Witten proposed a boundary CFT which is dual to the pure gravity on AdS_3 [1] (see also [2−8]). It is found that the partition function of boundary CFT has a nice interpretation as the sum over geometries in the bulk. However, there are some left-right asymmetric contributions in the partition functions which are difficult to interpret semi-classically. Moreover, the very existence of the pure gravity on AdS_3 as a quantum theory has not been established yet. Therefore, it is desirable to study AdS_3 gravity in the string theory setup. The obvious problem is that the dual CFT is not known in general. Even if the dual CFT is known, the partition function is usually hard to compute.

There are a few cases that we can study the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence quantitatively. In [9], Type IIB theory on $AdS_3 \times S^3 \times K3$ is studied by rewriting the partition function of BPS states (elliptic genus) as a sum over geometries, which is known as the Fareytail expansion. The difficulty appeared in the pure gravity on AdS_3 is avoided since the elliptic genus depends only on the left movers.

In this paper, we study the partition function of $\mathcal{N}=4$ $\mathrm{SU}(N)$ super Yang-Mills theory on K3. Via the string dualties, this is equal to the partition function of BPS states of N fundamental heterotic strings. Using the technique in [9, 10], we show that this partition function has an expansion as a sum over asymptotically AdS_3 geometries and argue that they are dual to a large number of heterotic strings. In section 2, we review the partition function of $\mathcal{N}=4$ SYM on K3 and its relation to the heterotic string. In section 3, we write down the Fareytail expansion of the partition function of $\mathcal{N}=4$ SYM on K3. In section 4, we discuss some questions.

2. $\mathcal{N} = 4$ SYM on K3 and heterotic strings: review

We first review the Vafa-Witten theory of topological $\mathcal{N} = 4$ SYM [11] and its relation to the BPS index of heterotic strings.

2.1 $\mathcal{N} = 4$ SYM on K3

In [11], it is shown that the topologically twisted SU(N) $\mathcal{N} = 4$ SYM on K3 computes the generating function of the Euler number of moduli space of k instantons

$$Z_N(\tau) = \sum_{k=0}^{\infty} q^{k-N} \chi\Big(\mathcal{M}_{N,k}(K3)\Big) \tag{2.1}$$

with $q = e^{2\pi i\tau}$. The $\mathcal{N} = 4$ SU(N) SYM with k instantons is realized by the following brane configuration in Type IIA theory:

$$N \text{ D4} \quad \text{on} \quad K3 \times \mathbb{R}_t \quad \oplus \quad k \text{ D0}\,, \tag{2.2}$$

where $\mathbb{R}_t$ denotes the time direction. In this brane picture, the shift $k \to k-N$ of instanton number in (2.1) is understood as the contribution of D0-brane charge from the curvature of K3.

The partition function (2.1) is evaluated as follows. Let us first consider the case of U(1) gauge theory. This is easily obtained by noting that the moduli space of U(1) instantons is equal to the Hilbert scheme of points on K3

$$\mathcal{M}_{1,k}(K3) = \mathrm{Hilb}^k(K3)\ . \tag{2.3}$$

It is well-known that the cohomology of this space is given by the Fock space of oscillators α^A_{-n} ($A = 1\cdots 24$) at level $L_0 = k$. Note that α^A_{-1} corresponds to the generator of $H^0(K3)\oplus H^2(K3) \oplus H^4(K3)$ and the higher modes α^A_{-n} ($n > 1$) correspond to the twisted sector of orbifold $(K3)^k/S_k$. From this representation, one finds that the partition function of U(1) theory is given by the partition function of 24 free bosons

$$G(\tau) = \frac{1}{\eta(\tau)^{24}}\ . \tag{2.4}$$

In the case of SU(N) theory, the partition function is given by an *almost* Hecke transform of the U(1) partition function $G(\tau)$ [12, 13]

$$Z_N(\tau) = \frac{1}{N^2} \sum_{ad=N, b\in\mathbb{Z}_d} d\, G\left(\frac{a\tau+b}{d}\right)\ . \tag{2.5}$$

When $N = p$ is prime, this expression simplifies to

$$Z_p(\tau) = \frac{1}{p^2} G(p\tau) + \frac{1}{p}\sum_{b=0}^{p-1} G\left(\frac{\tau+b}{p}\right)\ . \tag{2.6}$$

As discussed in [11, 12], the structure of summation in (2.5) can be physically understood by adding mass term to the adjoint scalar fields and breaking the theory to a factors of $\mathcal{N}=1$ SU(d) pure Yang-Mills. The summation over $b \in \mathbb{Z}_d$ comes from the d vacua of $\mathcal{N}=1$ SU(d) theory.

Note that $Z_N(\tau)$ itself is not a modular form, although $G(\tau)$ is a weight -12 modular form. This is related to the fact that the Montonen-Olive S-duality maps the SU(N) theory to a theory with different gauge group SU(N)/$\mathbb{Z}_N$. Therefore, Z_N does not come back to itself under the action of S-duality.

However, we can regard Z_N as a member of more general class of partition functions $Z_N^{(v)}$ with 't Hooft flux $v \in H^2(K3, \mathbb{Z}_N)$ turned on,[1] and identify $Z_N = Z_N^{(v=0)}$. The partition function with 't Hooft flux v is given by [14]

$$Z_N^{(v)}(\tau) = \frac{1}{N^2} \sum_{ad=N, b\in\mathbb{Z}_d} d\, G\left(\frac{a\tau+b}{d}\right) \delta_{dv,0} e^{-\pi i \frac{bv\cdot v}{aN}}, \tag{2.7}$$

where $v \cdot v' = \int_{K3} v \wedge v'$ is the intersection number. One can show that $Z_N^{(v)}$ transform as a vector-valued modular form of weight -12 [14]

$$Z_N^{(v)}(\gamma(\tau)) = (c\tau+d)^{-12} \sum_{v' \in H^2(K3,\mathbb{Z}_N)} M_{vv'}(\gamma) Z_N^{(v')}(\tau)\ . \tag{2.8}$$

Throughout this paper we use the usual notation for $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ and its action on τ

$$\begin{aligned} \gamma &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \\ \gamma(\tau) &= \frac{a\tau+b}{c\tau+d}\ . \end{aligned} \tag{2.9}$$

The modular matrix $M(\gamma)$ for $S = (\,0 \quad 1-1 \quad 0\,)$ and $T = (\,1 \quad 10 \quad 1\,)$ is given by

$$\begin{aligned} M_{vv'}(S) &= \frac{1}{N^{11}} e^{\frac{2\pi i}{N} v\cdot v'}, \\ M_{vv'}(T) &= \delta_{v,v'} e^{\frac{\pi i}{N} v\cdot v}\ . \end{aligned} \tag{2.10}$$

It is instructive to explicitly write down the first few terms of q-expansion of partition functions (2.4), (2.5)

$$\begin{aligned} G &= q^{-1} + 24 + 324q + 3200q^2 + 25650q^3 + \cdots, \\ Z_2 &= \frac{1}{4}q^{-2} + 30 + 3200q + 176337q^2 + 5930496q^3 + \cdots, \\ Z_3 &= \frac{1}{9}q^{-3} + \frac{80}{3} + 25650q + 5930496q^2 + 639249408q^3 + \cdots, \\ Z_4 &= \frac{1}{16}q^{-4} + \frac{63}{2} + 176256q + 143184800q^2 + 42189811200q^3 + \cdots\ . \end{aligned} \tag{2.11}$$

[1] One can introduce the theta series for the lattice $\Gamma^{3,19}$ by summing over the 't Hooft fluxes. This corresponds to considering U(N) gauge theory instead of SU(N) gauge theory [11].

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One immediately notices that Z_N has a 'gap' between q^{-N} and q^0, i.e., the coefficient of q^n vanishes in the range $-N+1 \leq n \leq -1$. This is true for general N:[2]

$$Z_N = \frac{1}{N^2} q^{-N} + 24 \sum_{a|N} \frac{1}{a^2} + \mathcal{O}(q) \ . \tag{2.12}$$

The existence of 'gap' is understood by counting the dimension of moduli space

$$\dim \mathcal{M}_{N,k}(K3) = 4N(k-N) + 4 \,, \tag{2.13}$$

which becomes negative when $k < N$. This implies that there is no contribution to Z_N from the instantons with the instanton number $k < N$.

2.2 Relation to heterotic strings

By the duality chain, we can dualize the D4-D0 configuration in (2.2) to a configuration in heterotic string theory. To see this, we first lift the IIA configuration (2.2) to the M-theory configuration:

$$N \text{ M5 \ on } \ K3 \times \mathbb{R}_t \times S^1_{\rm M} \quad \oplus \quad k \text{ units of momentum along } S^1_{\rm M} \ . \tag{2.14}$$

Here $S^1_{\rm M}$ denotes the M-theory circle in the eleventh direction. In order to relate this configuration to the topological $\mathcal{N}=4$ SYM, we perform a Wick rotation of the time direction $\mathbb{R}_t$ and compactify it to a thermal circle S^1_β. Then the worldvolume of M5-brane becomes $K3 \times T^2$ where $T^2 = S^1_\beta \times S^1_{\rm M}$. More generally, we replace the two-dimensional part of M5-brane worldvolume by a torus Σ_τ with an arbitrary modular parameter τ

$$\mathbb{R}_t \times S^1_{\rm M} \qquad \longrightarrow \qquad \text{Euclidean torus } \ \Sigma_\tau \ . \tag{2.15}$$

Using the relation between M5-brane compactified on a torus and $\mathcal{N}=4$ SYM, the moduli τ of torus Σ_τ is identified as the coupling constant of $\mathcal{N}=4$ SYM

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2_{\rm YM}} \ . \tag{2.16}$$

Finally, the relation between $\mathcal{N}=4$ SYM on K3 and the heterotic string follows from the identification of M5-brane wrapping around K3 and the fundamental heterotic string. Therefore, the M5-brane configuration (2.14) is dual to

$$N \text{ heterotic strings on } \Sigma_\tau \quad \oplus \quad k \text{ units of momentum along } S^1 \subset \Sigma_\tau \ . \tag{2.17}$$

In this heterotic string picture, the partition function Z_N is given by the index of BPS states (Dabholkar-Harvey states) in the $\mathcal{N}=(0,8)$ superconformal field theory of N fundamental heterotic strings. This is computed by setting the right-moving SUSY part to the ground state and summing over the left-moving bosonic side. For the single string case, this summation gives $\eta(\tau)^{-24}$, as expected from the result of U(1) $\mathcal{N}=4$ SYM (2.4). For $N>1$, the Hecke structure of SU(N) SYM partition function (2.5) is interpreted in the heterotic picture as the effect of multiple winding of genus one worldsheet around the target space torus Σ_τ [12, 16].

[2] Curiously, the q^0 term of Z_N is 24 times the integral of matrix model obtained by the dimensional reduction of $D=10$ super Yang-Mills to zero dimension [15].

3. Fareytail expansion of $\mathcal{N}=4$ SYM on K3

As discussed in [17, 18], a large number of coincident fundamental heterotic strings has a near horizon geometry of the form $AdS_3 \times M$, hence it is expected to have a holographic dual two-dimensional CFT. In the previous section, we saw that the partition function Z_N of $\mathcal{N}=4$ SYM on K3 captures the BPS spectrum of N fundamental heterotic strings. Therefore, it seems natural to identify Z_N as the BPS index of string theory on the AdS_3 dual of heterotic strings. Since we have Wick-rotated the time direction, the dual AdS_3 should be understood as the Euclidean AdS_3 and the torus Σ_τ is interpreted as the boundary of AdS_3. The modular parameter τ should be fixed as a boundary condition for the bulk metric.

The Euclidean AdS_3 is topologically a solid torus. There are many ways to fill inside the torus Σ_τ to make a solid torus. The bulk geometry is distinguished by the holomogy cycle of Σ_τ which becomes contractible. For instance, the spacial circle is contractible for the thermal AdS_3 and the temporal circle is contractible for the BTZ black hole.

To see the relation of the partition function Z_N to the bulk AdS_3 geometry[3], it is useful to rewrite Z_N as a Poincaré series. A general procedure is developed in [9, 10] and dubbed Fareytail expansion. The necessary ingredients are the modular matrix $M(\gamma)$ in (2.8) and the coefficient $c_v(n)$ of the polar part of $Z_N^{(v)} = \sum_n c_v(n) q^n$. Applying the general formula in [10] to our case, the Fareytail expansion of Z_N reads[4]

$$\sum_{\Gamma_\infty \backslash \Gamma} \equiv \lim_{K\to\infty} \sum_{(\Gamma_\infty\backslash\Gamma)_K} = \lim_{K\to\infty} \sum_{|c|\le K} \sum_{|d|\le K,(c,d)=1} . \tag{3.1}$$

$$\begin{aligned} Z_N(\tau) &= 12\sum_{a|N}\frac{1}{a^2} + \frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(c\tau+d)^{12}\sum_{v\in H^2(K3,\mathbb{Z}_N)} M^{-1}(\gamma)_{0v} \\ &\quad \times \sum_{n<0} c_v(n)\exp\left(2\pi i n\frac{a\tau+b}{c\tau+d}\right) R\left(\frac{2\pi i|n|}{c(c\tau+d)}\right), \end{aligned} \tag{3.2}$$

where $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, t\in\mathbb{Z} \right\}$ is the parabolic subgroup of $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, and $R(x)$ is defined by

$$R(x) = \frac{1}{(12)!}\int_0^x dt\, t^{12} e^{-t} . \tag{3.3}$$

In the large N limit, we expect that the expansion (3.2) can be interpreted as a sum over semi-classical geometries. One can see that in the large N limit the summation over 't Hooft flux is dominated by the $v=0$ term, since the leading term of $Z_N^{(v\neq 0)}$ is q^n with $n > -N$, while $Z_N^{(v=0)}$ starts with the term $\frac{1}{N^2}q^{-N}$. Therefore, in the the large N limit we can approximate Z_N as

$$Z_N \sim \frac{1}{2N^2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(c\tau+d)^{12} M^{-1}(\gamma)_{00}\exp\left(-2\pi i N\frac{a\tau+b}{c\tau+d}\right) R\left(\frac{2\pi i N}{c(c\tau+d)}\right) . \tag{3.4}$$

[3] The relation between the partition function $G(\tau)$ of U(1) theory and the black holes in $\mathcal{N}=4$ string theories is studied in [19]. The gravity dual of a single heterotic string is studied in [20].

[4] The sum over the coset $\Gamma_\infty\backslash\Gamma$ should be defined as a limit [10]

It seems natural to identify the exponential factor in (3.4) as the holomorphic part of the classical action of the $\mathrm{SL}(2,\mathbb{Z})$ family of BTZ black holes [21]

$$S = -4\pi N \,\mathrm{Im}\left(\frac{a\tau+b}{c\tau+d}\right) \,. \tag{3.5}$$

Namely, the partition function Z_N of $\mathcal{N}=4$ SYM on K3 admits a semi-classical expansion of sum over geometries in the AdS_3 background, which is holographically dual to heterotic strings. As we move τ on the upper half plane, the dominant term in the sum (3.4) changes. Since a large factor of N is multiplied in the classical action (3.5), this change of dominant contribution becomes a sharp phase transition in the large N limit. This is interpreted as the Hawking-Page transition [22] in the bulk gravity side. The phase diagram[5] is the same as that of the pure gravity on AdS_3 (figure 3b in [7]).

4. Discussion

In this paper, we studied the Fareytail expansion of the partition function of $\mathcal{N}=4$ SYM on K3 and interpreted it as a sum over geometries dual to fundamental heterotic strings. It is observed in [23] that the contribution of BTZ black hole is reproduced by taking the saddle point of instanton sum (2.1). To see this, recall that when the instanton number becomes large the Euler number of instanton moduli space scales as

$$\chi\Big(\mathcal{M}_{N,k}(K3)\Big) \sim e^{4\pi\sqrt{N(k-N)}} \qquad (k-N \gg 1)\,. \tag{4.1}$$

This essentially follows from the Cardy formula applied to the $c=24N$ CFT. Then the partition function (2.1) is approximated as

$$Z_N \sim \sum_k e^{4\pi\sqrt{N(k-N)}} q^{k-N}\,. \tag{4.2}$$

The saddle point $k=k_0$ of the above sum is given by

$$k_0 - N = -\frac{N}{\tau^2}\,, \tag{4.3}$$

and the value of the corresponding term turns out to be

$$e^{4\pi\sqrt{N(k_0-N)}} q^{k_0-N} = e^{2\pi i \frac{N}{\tau}}\,. \tag{4.4}$$

One can see that the exponent is nothing but the classical action of BTZ black hole. Therefore, it seems that the BTZ black hole corresponds to a condensate of large number of instantons. On the other hand, the zero-instanton term $\frac{1}{N^2}q^{-N}$ corresponds to the thermal AdS_3. It would be interesting to understand what happens when adding k_0 units of momentum to the fundamental heterotic string and see what triggers the phase transition in the heterotic string picture. It would also be interesting to study the zeros of $Z_N(\tau)$ and see if the Hawking-Page transition is associated with a condensation of Lee-Yang zeros [7]. Finally, it would be interesting to identify the $(c_L, c_R) = (24N, 12N)$ CFT of N fundamental heterotic strings.

[5] The phase diagram of $\mathcal{N}=4$ SYM on K3 was studied in [23]. However, the motivation of [23] seems to be different from ours.

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